(3) $Y(t) = \sqrt{t}\,X(t) - \int_0^t \frac{X(s)\,ds}{2\sqrt{s}}$

$F(t, x) = \sqrt{t}\,X(t)$ → Creating an SDE

$$dF = \frac{\partial F}{\partial x}\cdot dx + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\cdot (dx)^2 + \frac{\partial F}{\partial t}\cdot dt$$

$$= \sqrt{t}\,dx + 0\cdot(dx)^2 + \frac{1}{2\sqrt{t}}\cdot X(t)\,dt$$

$$dF = \sqrt{t}\,dX(t) + \frac{1}{2\sqrt{t}}\cdot X(t)\,dt$$

Integrating =

$$\int_0^t dF = \int_0^t \sqrt{s}\,dX(s) + \int_0^t \frac{1}{2\sqrt{s}}\,X(s)\,ds$$

$$\Rightarrow \int_0^t dF \Rightarrow \sqrt{t}\,X(t) - \sqrt{0} * X(0) = \sqrt{t}\,X(t)$$

hence $\sqrt{t}\,X(t) = \int_0^t \sqrt{s}\,dX(s) + \int_0^t \frac{1}{2\sqrt{s}}\,X(s)\,ds$

$$= \sqrt{t}\,X(t) - \int_0^t \frac{1}{2\sqrt{s}}\,X(s)\,ds = \int_0^t \sqrt{s}\,dX(s).$$

but $\sqrt{t}\,X(t) - \int_0^t \frac{1}{2\sqrt{s}}\,X(s)\,ds = Y(t)$

hence $Y(t) = \int_0^t \sqrt{s}\,dX(s)$ which is an Ito's integral. Hence process $Y(t)$ is a martingale.

B).

A) 1.

$$dF(S_1, S_2, \ldots, S_N) = \sum_{i=1}^{N} \frac{\partial F}{\partial S_i} \cdot dS_i + \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{\partial^2 F}{\partial S_i \partial S_j} \cdot (dS_i)(dS_j) + H.O.$$

$$= \sum_{i=1}^{N} \frac{\partial F}{\partial S_i} \cdot dS_i + \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=(i+1)}^{N} \frac{\partial^2 F}{\partial S_i \partial S_j} S_i S_j \sigma_i \sigma_j \rho_{ij} \, dt + H.O.$$

$$= \sum_{i=1}^{N} \frac{\partial F}{\partial S_i} (\mu_i S_i dt + S_i \sigma_i dX_i) + \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=(i+1)}^{N} \frac{\partial^2 F}{\partial S_i \partial S_j} (\sigma_i \sigma_j S_i S_j \rho_{ij} \, dt)$$

Now neglecting higher order terms →

$$dF = \left(\sum_{i=1}^{N} \frac{\partial F}{\partial S_i} \mu_i S_i + \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=(i+1)}^{N} \frac{\partial^2 F}{\partial S_i \partial S_j} (\sigma_i \sigma_j S_i S_j \rho_{ij}) \right) dt + \left(\sum_{i=1}^{N} \frac{\partial F}{\partial S_i} \cdot \sigma_i S_i dX_i \right)$$

Here we have a single drift term and N diffusion terms as can be seen from the previous equation.

(2) $Y(t) = e^{\sigma X - \frac{1}{2}\sigma^2 t}$

Apply Taylor series expansion and Ito's lemma

$$dY = \frac{\partial Y}{\partial X} \cdot dX + \frac{1}{2}\frac{\partial^2 Y}{\partial X^2} (dX)^2 + \frac{\partial Y}{\partial t} \cdot dt$$

$$\Rightarrow \sigma Y dX + \cancel{\frac{1}{2}(\sigma^2 Y) dt + \left(-\frac{1}{2}\sigma^2\right) Y dt}$$

$dY \Rightarrow \sigma Y dX$ hence proved

~~The term $Z(t)$ is $Y(t)$ and term~~

Term $Z(t) * g(t) = \sigma * \exp(\sigma X - \frac{1}{2}\sigma^2 t)$.

hence $Z(t)$ can be $\Rightarrow \sigma \exp(\sigma X)$

$$g(t) = \exp(-\tfrac{1}{2}\sigma^2 t)$$

The condition $Z(t) * g(t) = \sigma \exp(\sigma X - \frac{1}{2}\sigma^2 t)$ should be satisfied.